OCR A Further Maths AS-level

Pure Core

Formula Sheet

Provided in formula book	Not provided in formula book

Complex Numbers

The Language of Complex Numbers

Cartesian form of a complex number	$z = a + ib$, $a = Re(z), \quad b = Im(z)$
Modulus-argument form of a complex number	$z = a + bi, \quad \lvert z \rvert = r = \sqrt{a^2 + b^2}$, $\arg(z) = \theta = \tan^{-1}\frac{b}{a}$ $z = r(\cos\theta + i\sin\theta) = [r, \theta]$
Complex conjugate of a complex number	$z = a + ib$ has complex conjugate $z^* = a - bi$

Basic Operations

Multiplication in modulus-argument form	$z_1 z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$ $\lvert z_1 z_2 \rvert = \lvert z_1 \rvert \lvert z_2 \rvert, \ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
Division in modulus-argument form	$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$ $\left\lvert \frac{z_1}{z_2} \right\rvert = \frac{\lvert z_1 \rvert}{\lvert z_2 \rvert}, \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

Loci

Loci of points z such that $\lvert z - a \rvert = k$	Circle of radius k centred on $(Re(a), Im(a))$
Loci of points z such that $\lvert z - a \rvert = \lvert z - b \rvert$	Perpendicular bisector of the line from a to b
Loci of points z such that $\arg(z - a) = \alpha$	Half-line starting from a making an angle α with the real axis

Matrices

The Language of Matrices

An $m \times n$ matrix has m rows and n columns	$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$
The null matrix has zeros in every entry	$\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$
The identity matrix, I, is a square matrix with 1s on the leading diagonal and 0s elsewhere	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$
The transpose of a matrix $\boldsymbol{A}$, $\boldsymbol{A}^{\boldsymbol{T}}$, swaps the rows and columns of $\boldsymbol{A}$	$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

Addition and Multiplication

Addition and subtraction	$\begin{pmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \\ g_1 & h_1 & i_1 \end{pmatrix} \pm \begin{pmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \\ g_2 & h_2 & i_2 \end{pmatrix} = \begin{pmatrix} a_1 \pm a_2 & b_1 \pm b_2 & c_1 \pm c_2 \\ d_1 \pm d_2 & e_1 \pm e_2 & f_1 \pm f_2 \\ g_1 \pm g_2 & h_1 \pm h_2 & i_1 \pm i_2 \end{pmatrix}$
Scalar Multiplication	$k\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} ak & bk & ck \\ dk & ek & fk \\ gk & hk & ik \end{pmatrix}$
Matrix multiplication	$A: m \times n$ matrix, $B: n \times p$ matrix $(\boldsymbol{AB})_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ $\boldsymbol{AB}: n \times p$ matrix
Associativity and non-commutativity of matrix multiplication	$\boldsymbol{A}(\boldsymbol{B} \cdot \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ $\boldsymbol{AB} \neq \boldsymbol{BA}$ (In general. If this is true, $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to commute)

2D Linear Transformations

Transformation	Associated Matrix
Reflection in x axis.	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
Reflection in y axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
Enlargement by scale factor a	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$
Stretch parallel to x axis by scale factor a	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$
Stretch parallel to y axis by scale factor a	$\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$
Reflection in line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Reflection in line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
Anticlockwise rotation by an angle θ	$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$
Transformation with matrix $\boldsymbol{A}$ followed by transformation with matrix **B**	$\boldsymbol{BA}$

3D Rotations

The direction of positive rotation is taken to be anticlockwise when looking towards the origin from the positive side of the axis of rotation.

Rotation around x axis by an angle θ	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$
Rotation around y axis by an angle θ	$\begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$
Rotation around z axis by an angle θ	$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Invariance Under Transformations

Invariant point $\begin{pmatrix} x \\ y \end{pmatrix}$ under a transformation $\boldsymbol{M}$	$\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
Invariant line l	The image of any point on l is also on l

Determinants

Determinant of a 2 × 2 matrix	$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
Determinant of a matrix product	$\det \boldsymbol{AB} = \det \boldsymbol{A} \times \det \boldsymbol{B}$
Determinant of a multiple of a matrix	$\det(k\boldsymbol{A}) = k^2 \det(\boldsymbol{A})$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

Solutions of Simultaneous Equations

Condition for a system of equations $\boldsymbol{Mr} = \boldsymbol{a}$ to have a unique solution	$\det(\boldsymbol{M}) \neq 0$

Inverses of Matrices

Inverse matrix	$\boldsymbol{A}^{-1}$ is the inverse matrix of $\boldsymbol{A}$, such that $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$
Singular matrix	$\det(\boldsymbol{A}) = 0 \Rightarrow \boldsymbol{A}^{-1}$ does not exist. $\boldsymbol{A}$ is *singular*
Inverse of a 2×2 matrix	$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad ad - bc \neq 0$
Cofactor of an element – determinant of the matrix without the element's row and column	Cofactor of element a in $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is $\det\begin{pmatrix} e & f \\ h & i \end{pmatrix}$
Cofactor matrix of $\boldsymbol{A}$ – made of the cofactors of all elements of $\boldsymbol{A}$	Denoted by $\boldsymbol{C}$
Inverse of a 3×3 matrix	$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\boldsymbol{C}^T$
Inverse of a matrix product	$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
Inverse of a transformation	For a transformation given by matrix $\boldsymbol{M}$, its inverse is given by $\boldsymbol{M}^{-1}$

Further Vectors

Vector and Cartesian Forms of an Equation of a Straight Line

Vector equation of a line through the point $\boldsymbol{a}$ parallel to the vector $\boldsymbol{b}$	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
Cartesian equation of a line in 3D	For $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \lambda \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$, writing λ in terms of x, y and z: $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$

Scalar Product

Scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$	$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3 = \lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert\cos(\theta)$
Angle θ between two vectors $\boldsymbol{a}, \boldsymbol{b}$, or between two lines with these direction vectors	$\theta = cos^{-1}\left(\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\lvert\boldsymbol{a}\rvert\lvert\boldsymbol{b}\rvert}\right)$
Condition for $\boldsymbol{a}$ and $\boldsymbol{b}$ to be perpendicular vectors	$\boldsymbol{a} \cdot \boldsymbol{b} = 0$

Intersections of Lines

Intersection type	$\boldsymbol{r_1} = \boldsymbol{a_1} + \boldsymbol{\lambda_1 b_1}, \qquad \boldsymbol{r_2} = \boldsymbol{a_2} + \boldsymbol{\lambda_2 b_2}$
Parallel lines	$\boldsymbol{b_1} = \boldsymbol{\mu b_2}$
Intersecting lines	There exist values of λ_1 and λ_2 such that $\boldsymbol{r_1} = \boldsymbol{r_2}$
Skew	No such λ_1 and λ_2 as above exist

Vector Product

Vector Product – gives a vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$	$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_2b_3 - b_2a_3 \\ a_3b_1 - b_3a_1 \\ a_1b_2 - b_1a_2 \end{pmatrix}$

Further Algebra

Roots of Equations

Relationship between the roots and coefficients of a quadratic polynomial	Let p and q be roots of $ax^2 + bx + c = 0$. Then, $p + q = -\frac{b}{a}$, $pq = \frac{c}{a}$.
Relationship between the roots and coefficients of a cubic polynomial	Let p, q, and r be the roots of $ax^3 + bx^2 + cx + d = 0$. Then, $p + q + r = -\frac{b}{a}$, $pq + qr + rp = \frac{c}{a}$, $pqr = -\frac{d}{a}$.
Relationship between the roots and coefficients of a quartic polynomial	Let p, q, r and s be the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$. Then, $p + q + r + s = -\frac{b}{a}$, $pq + pr + ps + qr + qs + rs = \frac{c}{a}$, $pqr + pqs + prs + qrs = -\frac{d}{a}$, $pqrs = \frac{e}{a}$.

Transformations of Equations

Transformation of the roots of an equation, given a transformation of the equation	Let an equation in x have root $x = p$. Given a substitution $u = f(x)$, the transformed equation has a root $u = f(p)$